Indian Statistical Institute, Bangalore Centre
Solution set of Mid-semester Examination, 2014-15
Fourier Analysis

1. Let $\Phi_n,\ n = 1, 2, \ldots$ be a sequence of functions on $[-\pi, \pi]$ with the following properties: $\sup\{\int_{-\pi}^{\pi} |\Phi_n(t)|\, dt : n = 1, 2, \ldots\} < \infty$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_n(t)|\, dt \to 1$ as $n \to \infty$ and $\sup\{|\Phi_n(t)| : \pi \geq t \geq \delta\} \to 0$ as $n \to \infty$ for each $\delta \in (0, \pi)$. If $f \in L^1[-\pi, \pi]$, then prove that $f * \Phi_n \to f$ in $L^1[-\pi, \pi]$.

Proof. First we claim that $\lim\limits_{n\to\infty} \|\Phi_n * f - f\|_\infty = 0$ for all continuous f on $[-\pi, \pi]$. Observe that

$$(\Phi_n * f)(x) - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_n(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_n(t)[f(x-t) - f(x)]\, dt$$

Take $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_n(t)\, dt$ and the translation operator $(T_t f)(x) = f(x - t)$. Since continuous function f on $[-\pi, \pi]$ is uniformly continuous, given $\epsilon > 0$ there exists $\delta \in (0, \pi]$ such that $\|T_t f - f\|_\infty = \sup\{|f(x-t) - f(x)| : x \in [-\pi, \pi]\} < \epsilon$ whenever $|t| \leq \delta$. Then

$$\begin{aligned}\|\Phi_n * f - c_n f\|_\infty &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_n(t)|\|T_t f - f\|_\infty\, dt \\ &= \frac{1}{2\pi}\Big[\int_{|t|\leq\delta} |\Phi_n(t)|\|T_t f - f\|_\infty\, dt + \int_{\delta\leq|t|\leq\pi} |\Phi_n(t)|\|T_t f - f\|_\infty\, dt\Big] \\ &\leq M\epsilon \quad \text{(for some constant } M)\end{aligned}$$

where we have used the given facts $\sup\{\int_{-\pi}^{\pi} |\Phi_n(t)|\, dt : n = 1, 2, \ldots\} < \infty$ and $\sup\{|\Phi_n(t)| : \pi \geq t \geq \delta\} \to 0$ as $n \to \infty$ for each $\delta \in (0, \pi)$. Thus $\limsup_n \|\Phi_n * f - c_n f\|_\infty \leq M\epsilon$. Since $\epsilon > 0$ is arbitrary and $c_n \to 1$, we have $\lim_n \|\Phi_n * f - c_n f\|_\infty = 0$.

Let $f \in L^1[-\pi, \pi]$. Then given $\epsilon > 0$ there exists a continuous function g on $[-\pi, \pi]$ such that $\|f - g\|_1 < \epsilon$. Recall that $L^1[-\pi, \pi]$ is a Banach algebra w.r.t. multiplication as convolution. Therefore for all $n \in \mathbb{N}$

$$\|\Phi_n * f - \Phi_n * g\|_1 \leq \|\Phi_n\|_1 \|f - g\|_1 \leq M\epsilon$$

Since g is continuous on $[-\pi, \pi]$, we have $\|\Phi_n * g - g\|_\infty < \epsilon$ for sufficiently large n as we have proved in the above discussion. Then

$$\|\Phi_n * g - g\|_1 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |(\Phi_n * g - g)(t)| \, dt \leq \|\Phi_n * g - g\|_\infty \leq \epsilon$$

for sufficiently large n. Hence

$$\begin{aligned}\|\Phi_n * f - f\|_1 &= \|(\Phi_n * f - \Phi_n * g) + (\Phi_n * g - g) + (g - f)\|_1 \\ &\leq \|\Phi_n * f - \Phi_n * g\|_1 + \|\Phi_n * g - g\|_1 + \|f - g\|_1 \leq (M + 2)\epsilon\end{aligned}$$

for sufficiently large n. Thus $\Phi_n * f = f * \Phi_n \to f$ in $L^1[-\pi, \pi]$. □

2. If $f \in L^2[-\pi, \pi]$ and $f_k = f * f * \ldots * f$ (k-fold convolution of f), then show that $\|f_k\|_2^{\frac{1}{k}} \to \sup\{|f(n)| : n \geq 1\}$ as $k \to \infty$.

Proof. Observe that

$$\|f_k\|_2^{1/k} = \|(\hat{f})^k\|_2^{1/k} = \|(\hat{f})^2\|_k^{1/2} \to \|(\hat{f})^2\|_\infty^{1/2}$$

where we have used norm equivalence in L^2, properties of transforms of convolutions, and the fact that $\|x\|_k \to \|x\|_\infty$ as $k \to \infty$ for $x \in \ell^k$. □

3. Prove or disprove: if f is periodic function of bounded variation then the Fourier series of f converges to f uniformly.

Proof. We prove that above statement is not always true by the following example: Consider the 2π-periodic extension of $f(t) = \dfrac{t}{2}$ on $[-\pi, \pi]$. It is integrable, and of bounded variation on $[-\pi, \pi]$. Then for any $t \in (-\pi, \pi)$ we have

$$f(t) = \frac{t}{2} = \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n} \sin nt.$$

The series converges to 0 for $t = -\pi, \pi$ (By Jordon's point-wise convergence theorem for the function f its Fourier series converges to $\dfrac{1}{2}[f(t^-) + f(t^+)]$ at any point $t \in [-\pi, \pi]$). Thus Fourier series of f does not converge to f uniformly. □

4. If f and g are absolutely continuous on $[a, b]$ show that fg is also absolutely continuous. Use this to show that

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'\,dx.$$

Proof. Since f, g are continuous, they are bounded on [a, b], say $|f(x)| \leq M$ and $|g(x)| \leq N$ for all $x \in [a, b]$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that if $\{(a_i, b_i)\}$ is finite disjoint collection of open intervals in $[a, b]$ with $\sum(b_i - a_i) < \delta_1$, then

$$\sum |f(b_i) - f(a_i)| < \frac{\epsilon}{2M}.$$

Similarly, there exists $\delta_2 > 0$ such that if $\{(a_i, b_i)\}$ is finite disjoint collection of open intervals in $[a, b]$ with $\sum(b_i - a_i) < \delta_1$, then

$$\sum |g(b_i) - g(a_i)| < \frac{\epsilon}{2N}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned}\sum |f(b_i)g(b_i) - f(a_i)g(a_i)| &\leq \sum [|f(b_i)g(b_i) - f(b_i)g(a_i)| + |f(b_i)g(a_i) - f(a_i)g(a_i)|] \\ &\leq M \sum |f(b_i) - f(a_i)| + N \sum |g(b_i) - g(a_i)| < \epsilon.\end{aligned}$$

Hence fg is absolutely continuous.

Since f, g, fg are absolutely continuous functions, then $f',\ g',\ (fg)'$ exist a.e. on $[a, b]$ and for all $x \in [a, b]$ we have

$$\begin{aligned}(fg)(x) &= \int_a^x (fg)'(x)\,dx + (fg)(a) \\ &= \int_a^x f'(x)g(x)\,dx + \int_a^x f(x)g'(x) + (fg)(a) \\ &= \int_a^x f'(x)g(x)\,dx + \int_a^x f(x)g'(x) + (fg)(a)\end{aligned}$$

By putting $x = b$ we get,

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx. \qquad \square$$

5. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$ for $0 \leq x \leq 2\pi$ and show that the formula does not hold for any $x \in \mathbb{R} \setminus [0, 2\pi]$.

Proof. Take $f(x) = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$. By Fourier coefficients formula, we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi} [\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}]\,dx = 0. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi} [\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}]\cos nx\,dx = \frac{1}{n^2}. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi} [\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}]\sin nx\,dx = 0. \end{aligned}$$

where we have used integration by parts for the formula of a_n, b_n for each $n \geq 1$. Hence

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \quad \text{for} \quad 0 \leq x \leq 2\pi.$$

Suppose if possible formula holds for some $x_0 \ \in \mathbb{R} \setminus [0, 2\pi]$. Then there exists $n \in \mathbb{Z}$ such that $x_0 \in [2n\pi, 2(n+1)\pi]$ where $n \neq 0$. By periodicity of the Fourier series we have

$$\begin{aligned} & f(x_0 - 2n\pi) = f(x_0) \\ & \Rightarrow \frac{(x_0 - 2n\pi)^2}{4} - \frac{\pi(x_0 - 2n\pi)}{2} + \frac{\pi^2}{6} = \frac{x_0^2}{4} - \frac{\pi x_0}{2} + \frac{\pi^2}{6} \\ & \Rightarrow x_0 = (n+1)\pi \end{aligned}$$

which is a contradiction as $x_0 \in [2n\pi, 2(n+1)\pi]$ with $n \neq 0$. Thus the formula does not hold for any $x \in \mathbb{R} \setminus [0, 2\pi]$. □

6. Let $\{c_n\}$ be a sequence of positive numbers decreasing to 0. For what value of x does the series

$$\sum_{n=1}^{\infty} c_n \sin(nx)$$

converge? Justify your claim.

Proof. Define $S_n(x) = \sum_{m=1}^{n} e^{imx} = \frac{e^{inx}-1}{1-e^{-ix}}, \quad x \neq k\pi, k \in \mathbb{Z}$. Then

$$|S_n(x)| \leq \frac{2}{|1-e^{-ix}|} = \frac{1}{|\sin\frac{x}{2}|}.$$

Now using a technique for infinite series which is analogous to integration by parts for all $n \in \mathbb{N} \cup \{0\}, p \in \mathbb{N}$:

$$\begin{aligned}\sum_{m=n+1}^{n+p} c_m e^{imx} &= e^{inx}\big[c_{n+1}S_1(x) + \sum_{m=2}^{p} c_{n+m}(S_m(x) - S_{m-1}(x))\big] \\ &= e^{inx}\big[\sum_{m=1}^{p-1} S_m(x)(c_{n+m} - c_{n+m+1}) + c_{n+p}S_p(x)\big]\end{aligned}$$

Therefore

$$\big|\sum_{m=n+1}^{n+p} c_m e^{imx}\big| \leq \frac{1}{|\sin\frac{x}{2}|}\big(\sum_{m=1}^{p-1}(c_{n+m} - c_{n+m+1}) + c_{n+p}\big) = \frac{c_{n+1}}{|\sin\frac{x}{2}|}$$

where we have used $c_{n+m} \geq 0,\ c_{n+m} - c_{n+m+1} \geq 0$. From this estimate, it follows that series $\sum_{m=1}^{\infty} c_m e^{imx}$ converges for $x \neq k\pi,\ k \in \mathbb{Z}$. Therefore, its imaginary part

$$\sum_{n=1}^{\infty} c_n \sin(nx)$$

converges for $x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$ and for $x = k\pi, k \in \mathbb{Z}$ series converges to 0 trivially. So given series converges for all $x \in \mathbb{R}$. $\square$